LA-UR-04-7151

Approved for public release; distribution is unlimited.

Title: BELIEF FUNCTIONS AND PROBABILISTIC THINKING

Author(s): ALYSON G. WILSON, LANL
KARI SENTZ, LANL

Submitted to: The American Statistician

Los Alamos
NATIONAL LABORATORY

Belief Functions and Probabilistic Thinking

Alyson Wilson and Kari Sentz*

September 26, 2004

* **Alyson Wilson** is a Technical Staff Member and the Technical Lead for DoD Programs in the Statistical Sciences Group at Los Alamos National Laboratory, P.O. Box 1663, MS F600 Los Alamos, NM 87545 (email: *agw@lanl.gov*). Kari Sentz is a Graduate Research Assistant in the Statistical Sciences Group at Los Alamos National Laboratory and a doctoral candidate in Systems Science at Binghamton University (email: *ksentz@lanl.gov*). This work has benefited greatly from extensive conversations with Sallie Keller-McNulty, Nozer Singpurwalla, Bill Oberkampf, Jon Helton, Scott Ferson, and Art Dempster.

Abstract

This paper develops the arguments for the use of belief functions (also known as *Dempster-Shafer methods*) in probabilistic modeling. We discuss the motivation for belief functions through examples. Then we outline the basic properties of belief functions. From this we develop a simple example in reliability.

KEY WORDS: Dempster-Shafer methods; belief functions; random set; uncertainty quantification; decision-making.

1 Motivation

Given a fair die, what is the probability of rolling a 6? Clearly, 1 in 6. Suppose, however, that instead of knowing the die is fair, you know that the probability of rolling an even number is the same as rolling the probability of an odd number. What is the probability of rolling a 6?

There are a number of ways to think about this problem. One way is to employ the *Principle of Insufficient Reason*, which says that if there is no reason to choose between several alternatives, they should be treated as equally likely (Hacking 2001). In this case, this would imply that the probability of rolling a 6 is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

Using the information that "the probability of evens is the same as the probability of odds" plus Principle of Insufficient Reason leads to the same answer as assuming the die is fair. Clearly, using the Principle of Insufficient Reason adds information. What can we say if we choose not to add any additional information?

We can say that the probability of rolling a 6 is at least 0 and at most $\frac{1}{2}$. Six is an even number, so the probability of rolling a 6 can be at most the probability of rolling an even number, so it is bounded at $\frac{1}{2}$. However, we also have no guarantee that a 6 will ever appear, so the probability could be 0.

Extending this logic, there are a set of probability distributions describing a roll of the die that are consistent with the information "the probability of evens is the same as the probability of odds." Letting $p_i = P(\text{roll } = i)$, any distribution with $0 \leq p_i \leq \frac{1}{2}$, $\sum_{i=1,3,5} p_i = \frac{1}{2}$, and $\sum_{i=2,4,6} p_i = \frac{1}{2}$ is valid. In particular, $p_i = \frac{1}{6}, i = 1, \ldots, 6$ works, as does $p_1 = \frac{1}{2}, p_4 = \frac{1}{2}$.

2 Fundamental Ideas

The previous example can be described using the theory of belief functions (also known as evidence theory and Dempster-Shafer theory). The basic ideas and properties of belief function methods are derived from the original notion of the multi-valued map from Dempster (1967). For simplicity, this discussion focuses on finite sets; for formal treatments of infinite sets, see Kohlas and Monney (1995) and Kramosil (2001).

2.1 Probability

Recall the standard theoretical framework used for probability (Billingsley 1986). Start with a set of "events" or "outcomes" (the *sample space* or *universal set*) and define a measure with certain properties on subsets of the original set.

The mathematics is as follows. Start with a finite set Θ; for concreteness, let $\Theta = \{R, Y, G\}$. Let S_Θ be a sigma field generated by Θ. A class S_Θ of subsets of Θ is called a *field* if it contains Θ itself and is closed under the formation of complements and finite unions. In symbols,

1. $\Theta \in S_\Theta$
2. $A \in S_\Theta$ implies $A^c \in S_\Theta$
3. $A, B \in S_\Theta$ implies $A \cup B \in S_\Theta$.

A class S_Θ of subsets of Θ is called a *sigma field* if it is a field and also closed under the formation of countable unions. (For a finite universal set, this does not add any conditions to the definition of a field.) Again for concreteness, let $S_\Theta = 2^\Theta = \{\emptyset, \{R\}, \{Y\}, \{G\}, \{R, Y\}, \{R, G\}, \{Y, G\}, \{R, Y, G\}\}$, the power set (set of all subsets) of Θ. 2^Θ is not the only sigma field that can be constructed from Θ: consider $\{\emptyset, \{R\}, \{Y, G\}, \{R, Y, G\}\}$.

We have a universal set Θ and a sigma field S_Θ. Now we want to define a probability measure on S_θ. A *set function* is a real-valued function defined on some class of subsets of Θ. A set function μ on a field S_Θ is a *probability measure* if it satisfies the following conditions:

1. $0 \le \mu(A) \le 1$ for $A \in S_\theta$
2. $\mu(\emptyset) = 0, \mu(\Theta) = 1$
3. if $A_1, A_2, \ldots$ are disjoint with $A_i \in S_\Theta$ and $\cup_{i=1}^\infty A_i \in S_\Theta$, then $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$

For concreteness, define

$$\begin{aligned} \mu(\emptyset) &= 0, & \mu(\{R,Y\}) &= 0.65, \\ \mu(\{R\}) &= 0.25, & \mu(\{R,G\}) &= 0.60, \\ \mu(\{Y\}) &= 0.40, & \mu(\{Y,G\}) &= 0.75, \\ \mu(\{G\}) &= 0.35, & \mu(\{R,Y,G\}) &= 1. \end{aligned}$$

The triple (Θ, S_Θ, μ) is called a *probability space*, with universal set Θ, sigma field S_Θ, and probability measure μ.

2.2 Multi-Valued Map and Belief Functions

Suppose that in addition to our probability space (Θ, S_Θ, μ) we also have another *measurable space* defined by a finite set Ω and a sigma field S_Ω. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $S_\Omega = 2^\Omega$. The elements of S_Ω are given in Table 1. Also define a second measurable space, with $S_\Omega = 2^\Omega$ as the universal set and S_{2^Ω} as the sigma field.

Now define a map $\Gamma : \Theta \to S_\Omega \backslash \emptyset$. For concreteness, define $\Gamma(R) \to \{1, 3, 5\}$, $\Gamma(Y) \to \{2, 4, 6\}$, $\Gamma(G) \to \{1, 2, 3\}$. We can use the map Γ to define a probability π on the measurable space (S_Ω, S_{2^Ω}). Define $\pi(E \in S_{2^\Omega}) = \mu(\{\theta \in \Theta : \Gamma(\theta) \in E\})$. In our example, $\pi(\{\{1\}, \{1, 2, 3\}\}) = \mu(\{G\}) = 0.35$ and $\pi(\{\{2, 4, 6\}, \{1, 3, 5\}\}) = \mu(\{R, Y\}) = 0.65$. Calculating π for each element of S_{2^Ω} and verifying that it is a probability measure are left as exercises.

Notice that this definition of the induced probability π on (S_Ω, S_{2^Ω}) parallels that of a random variable, except that the map Γ is a one-to-many map from $\Theta \to S_\Omega \backslash \emptyset$ instead of a one-to-one or many-to-one map from $\Theta \to \Omega = \mathbb{R}$. The map Γ is called a *set-valued random variable* or *random set* (Wasserman 1990).

How does this map back to the original example with our die? Let $\Theta = \{\text{even, odd}\}$ and $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let μ be a probability measure on S_Θ such that $\mu(\emptyset) = 0$, $\mu(\{\text{even}\}) = 0.5$, $\mu(\{\text{odd}\}) = 0.5$, and $\mu(\{\text{even, odd}\}) = 1$. Define $\Gamma : \Theta \to S_\Omega \backslash \emptyset$ as $\Gamma(\text{even}) \to \{2, 4, 6\}$ and $\Gamma(\text{odd}) \to \{1, 3, 5\}$. Γ induces a probability distribution on (S_Ω, S_{2^Ω}).

However, think back to our original question, which was to find the probability of rolling a 6. To answer this question requires a probability distribution (or set of distributions, as we saw earlier) on (Ω, S_Ω), not on (S_Ω, S_{2^Ω}). In particular, we would like to know the probability of $\{6\} \in S_\Omega$. How do we use the induced probability distribution on (S_Ω, S_{2^Ω}) to talk about (Ω, S_Ω)?

To do this, we introduce the ideas of the *basic probability assignment*, *belief function*, and *plausability function*. Notice that even in small problems, like ours here, it can be difficult to work with the probability distribution induced on (S_Ω, S_{2^Ω}) either because of the size of S_{2^Ω} or because of the intuitive problems of working with distributions on sets of sets. Also, the question of interest, as with our example, may be about Ω.

Consider the function $m : S_\Omega \to [0, 1]$ defined as $m(F \in S_\Omega) = \mu(\{\theta \in \Theta : \Gamma(\theta) = F\})$. m assigns to each element in S_Ω the mass of any element θ that maps directly to it through Γ. The function m is usually called the *basic probability assignment* or *b.p.a.* on (Ω, S_Ω) induced by Γ. It would be convenient if m always defined a probability distribution on (Ω, S_Ω), but frequently it does not. It will define a probability distribution on when (Ω, S_Ω) when m is assigned to only singletons. This is the special case in belief functions when the basic probability assignment collapses to a probability in the classical sense. Although the b.p.a. is not a probability distribution on Ω, it does have the properties that $m(\emptyset) = 0$ and $\sum_{F \in S_\Omega} m(F) = 1$. In our die example, the b.p.a. is given in Table 2.

The basic probability assignment can be used to construct other set functions on S_Ω. In particular, define the following two functions from $S_\Omega \to [0, 1]$. For $F, G \in S_\Omega$:

$$\begin{aligned} Bel(F) &= \sum_{G \subseteq F} m(G) \\ Pl(F) &= \sum_{G \cap F \neq \emptyset} m(G). \end{aligned}$$

$Bel()$ is called the *belief function* induced by Γ on S_Ω and $Pl()$ is called the *plausability function.* (Shafer 1976) $Bel()$ and $Pl()$ can also be written in terms of Γ and μ; in particular,

$$\begin{aligned} Bel(F) &= \mu(\{\theta \in \Theta : \Gamma(\theta) \subseteq F\}) \qquad (1) \\ Pl(F) &= \mu(\{\theta \in \Theta : \Gamma(\theta) \cap F \neq \emptyset\}). \end{aligned}$$

It can be shown that $Bel(F) = 1 - Pl(F^c)$. In addition, $Bel()$ and $Pl()$ are probability measures if and only if $Bel(F) = Pl(F)$ for all $F \in S_\Omega$.

Heuristically, belief is the amount of evidence directly supporting F—the amount of mass from the basic probability assignment completely contained in F. Plausability is the amount of evidence consistent with F—the amount of mass that intersects F.

The belief and plausability functions for our die example are given in Table 3 and 4. Notice that $Bel(\{6\}) = 0$ and $Pl(\{6\}) = 0.5$. Recall the earlier heuristic argument about the smallest and largest probability of rolling a 6 given that the probability of rolling an even number is 0.5. (*We can say that the probability of rolling a 6 is at least* 0 *and at most* $\frac{1}{2}$. *Six is an even number, so the probability of rolling a 6 can be at most the probability of rolling an even number, so it is bounded at* $\frac{1}{2}$. *However, we also have no guarantee that a 6 will ever appear, so the probability could be* 0.) The belief and plausability functions are the mathematical formalization of these arguments.

Any probability measure π that assigns mass to (Ω, S_Ω) such that for each $F \in S_\Omega, Bel(F) \leq \pi(F) \leq Pl(F)$ is *compatible* with the available information.

2.3 Betting and Coherence

Belief functions, plausability functions, and basic probability assignments are not probability measures. However, they can be used to define sets of distributions compatible with the available information. When uncertainty is described in terms of a probability measure, there is a betting interpretation. In particular, suppose that I assign probability p_E to an event E. The betting interpretation is a two-sided bet. I am willing to bet $\$p_E$ to get \$1 if E occurs or I am willing to bet $\$(1-p_E)$ to get \$1 if E does not occur.

Belief and plausability also have a betting interpretation. In particular, if $Bel(E)$ is my belief function for event E and $Pl(E)$ is my plausability function for event E, then I am willing to bet at most $\$Bel(E)$ to get \$1 if E occurs, and I am willing to bet at least $\$Pl(E)$ to get \$1 if E does not occur. These bets are summarized in Figure 1 (de Cooman and Zaffalon, 2004).

The betting interpretation of probability, belief, and plausability operationalizes the mathematical definitions and makes it easy to check certain properties. For example, one desirable property of an uncertainty quantification is *avoiding sure loss*. This means that there is not a finite collection of bets that I am willing to accept that will guarantee that I lose money. For example, consider the following two bets: To avoid sure loss,

$$\begin{array}{ll}
\text{on } A: & I_A - Bel(A) \\
\text{on } A^c: & I_{A^c} - Bel(A^c) \\
\hline
\text{together:} & 1-(Bel(A)+Bel(A^c))
\end{array}$$

$$\begin{aligned}
1-(Bel(A)+Bel(A^c)) &\geq 0 \\
Pl(A)-Bel(A) &\geq 0 \\
Pl(A) &\geq Bel(A).
\end{aligned}$$

This property is easy to check from Equation 1.

3 Reliability Example

Now consider the application of belief function methods to a reliability problem. Suppose that we have an item, and we are interested in assessing our uncertainty about whether the item will survive to time t or will fail by time t. We denote these two possibilities using a binary variable X, with $X = 1$ if the item survives, and $X = 0$ if the item fails.

Unfortunately, we are cannot test the item directly. Instead we can observe the presence or the absence of a detectable anomaly. An anomaly could be a visible defect, or noticeable damage, or some other suitable indicator of imperfection. Anomalies could be present and yet not be detected. We denote the presence of a detected anomaly by letting a binary variable Y take the value 1; the absence of a detectable anomaly by letting $Y = 0$. The presence of an anomaly does not necessarily imply that $X = 0$; however, for simplicity, we assume that the absence of an anomaly implies that $X = 1$. Figure 2 summarizes the scenario.

Now consider the application of Dempster-Shafer methods to this "indirect" data reliability problem. Let $p_a = P(Y = 1)$, the probability of an anomaly. Figure 2 defines a multi-valued map $\Gamma(\text{anomaly}) = \{\text{survives to t, fails before t}\}$ and $\Gamma(\text{no anomaly}) = \{\text{survives to t}\}$. This induces the b.p.a., belief, and plausability functions given in Table 5.

Any probability measure that assigns $P(X = 1)$ in $[1 - p_a, 1]$ and $P(X = 0)$ in $[0, p_a]$ is compatible with the information. There are essentially two schools of interpretation for belief and plausability functions. One interpretation is as the "lower and upper bounds for some unknown probability distribution" (Kohlas and Monney 1995). This is the interpretation we have been using. The other interpretation of Dempster-Shafer methods is in terms of the "degree of support" for a hypothesis (Shafer 1976). A degree of support of at least $1 - p_a$ is assigned to survival, as any observations of "no anomaly" certainly support survival. However, there is never unequivocal evidence of failure–even in the presence of an anomaly, the part may survive until time t. Since there is no evidence inconsistent with survival until time t, the maximum degree of support is 1.

Notice that in a traditional Bayesian reliability analysis, either one distribution compatible with the given information would be selected, or a distribution would be placed over the possible distributions. In the belief function approach, there is simply a set of distributions compatible with the given information.

4 Decision Theory

One consequence of the belief function approach to specifying uncertainty is that classical decision theory must be reconsidered. The classical formulation of decision-making under uncertainty is as follows. Let Ω be a set of "states of nature" (the universal set); let A be a set of "actions"; let $U(\omega, a)$ be the utility that is received of action $a \in A$ is taken when state of nature $\omega \in \Omega$ holds; let μ be a probability measure that captures my uncertainty about which state of nature ($\omega \in \Omega$) holds. Classical decision theory chooses the action $a \in A$ that maximizes the expected utility U with respect to μ.

If uncertainty about the states of nature is captured using belief functions, there are a set of probabilities that describe our uncertainty about μ. In this case, there is no guarantee that there will be an action that maximizes expected utility. Consider the reliability example from Section 3 and the states of nature, actions, and utilities given in Table 6. Let $P(X = 1) = P(\text{survives to time } t) = p_S$. The expected utility of using the item at time t is $2p_S - 5(1-p_S) = 7p_S - 5$. The expected utility of not using the item at time t is $-p_S + (1-p_S) = 1-2p_S$. The expected utility of using the item at time t is greater than the expected utility of not using the item at time t if $7p_S - 5 > 1 - 2p_S$ or $p_S > \frac{2}{3}$. Recall that to be compatible with the given information, $p_S = P(X = 1)$ must be in $[1 - p_a, 1]$. So if $p_a \leq \frac{1}{3}$, then using the item at time t maximizes the expected utility and should be the action of choice. However, if $p_a > \frac{1}{3}$, then neither action uniquely maximizes the expected utility.

5 Discussion

The reliability example considered here is representative of the class of problems in reliability which is receptive to a belief function representation. Belief functions present a particular advantage in the case where we are interested in one phenomenon, but that we can only collect data about another, related phenomenon. For example, we are interested in the reliability of system components but only have data on the system itself. In other words, belief functions may be useful when we have "indirect" data or more general data about the phenomenon of interest. We can relate the two phenomena through a set-valued map. Although the data that we can collect does not uniquely determine the outcome of interest, there is still useful information that can be exploited for the purposes of an analysis.

Belief functions are only one of many possible uncertainty representations. Traditional probabilistic methods can be considered a special case of the more general representation that the use of belief functions affords. It is important to note that there are uncertainty representations that are more general than belief functions. These are often referred to as *imprecise probabilities* (Walley 1991). While the applications of imprecise probabilities are only at the beginning of their development, we consider this as an important area for future research in reliability.

References

[1] Billingsley, P. (1986), *Probability and Measure*, 2nd Edition, New York: John Wiley & Sons.

[2] de Cooman, G., and Zaffalon, M. (2004), "Updating Beliefs with Incomplete Observations," *Artificial Intelligence*, To Appear.

[3] Dempster, A. (1967), "Upper and lower probabilities induced by a multi-valued mapping," *Annals of Mathematical Statistics*, 38, 2, 325–339.

[4] Hacking, I. (2001), *An Introduction to Probability and Inductive Logic,* Cambridge, UK: Cambridge University Press.

[5] Kohlas J. and Monney, P. (1995), *A Mathematical Theory of Hints: An Approach to the Dempster-Shafer Theory of Evidence,* Berlin: Springer-Verlag.

[6] Kramosil, I. (2001), *Probabilistic Analysis of Belief Functions*, New York: Plenum Press.

[7] Shafer, G. (1976), *A Mathematical Theory of Evidence,* Princeton, NJ: Princeton University Press.

[8] Walley, P. (1991), *Statistical Reasoning with Imprecise Probabilities*, London: Chapman and Hall.

[9] Wasserman, L. (1990), "Belief functions and statistical inference," *Canadian Journal of Statistics*, 18, 3, 183–196.

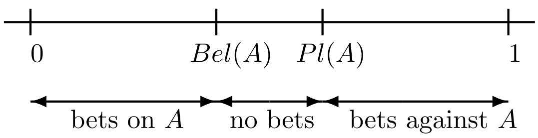

Figure 1: Belief, Plausability, and Betting

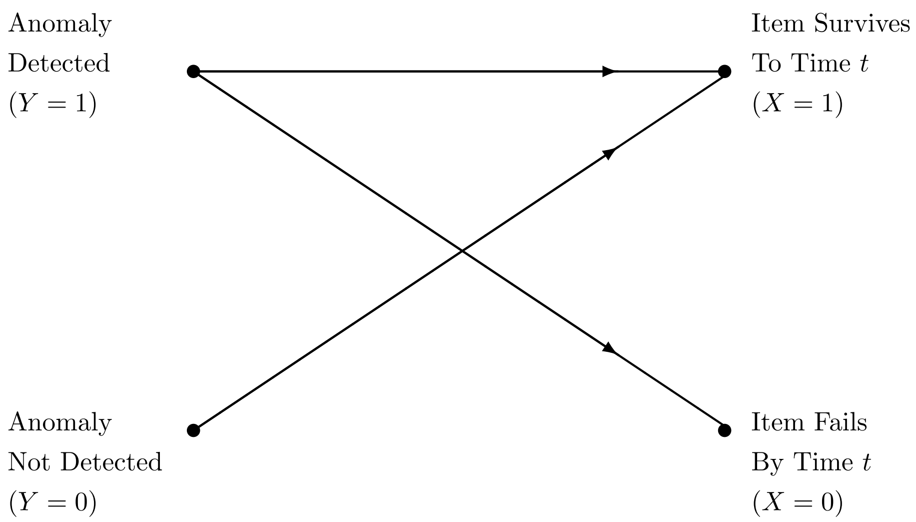

Figure 2: Effect of Anomalies on Survival

Table 1: Elements of Power Set, S_Ω, for Die Example

$\emptyset$	$\{1,2,3\}$	$\{1,2,3,5\}$
$\{1\}$	$\{1,2,4\}$	$\{1,2,3,6\}$
$\{2\}$	$\{1,2,5\}$	$\{1,2,4,5\}$
$\{3\}$	$\{1,2,6\}$	$\{1,2,4,6\}$
$\{4\}$	$\{1,3,4\}$	$\{1,2,5,6\}$
$\{5\}$	$\{1,3,5\}$	$\{1,3,4,5\}$
$\{6\}$	$\{1,3,6\}$	$\{1,3,4,6\}$
$\{1,2\}$	$\{1,4,5\}$	$\{1,3,5,6\}$
$\{1,3\}$	$\{1,4,6\}$	$\{1,4,5,6\}$
$\{1,4\}$	$\{1,5,6\}$	$\{2,3,4,5\}$
$\{1,5\}$	$\{2,3,4\}$	$\{2,3,4,6\}$
$\{1,6\}$	$\{2,3,5\}$	$\{2,3,5,6\}$
$\{2,3\}$	$\{2,3,6\}$	$\{2,4,5,6\}$
$\{2,4\}$	$\{2,4,5\}$	$\{3,4,5,6\}$
$\{2,5\}$	$\{2,4,6\}$	$\{1,2,3,4,5\}$
$\{2,6\}$	$\{2,5,6\}$	$\{1,2,3,4,6\}$
$\{3,4\}$	$\{3,4,5\}$	$\{1,2,4,5,6\}$
$\{3,5\}$	$\{3,4,6\}$	$\{1,2,3,5,6\}$
$\{3,6\}$	$\{3,5,6\}$	$\{1,3,4,5,6\}$
$\{4,5\}$	$\{4,5,6\}$	$\{2,3,4,5,6\}$
$\{4,6\}$	$\{1,2,3,4\}$	$\{1,2,3,4,5,6\}$
$\{5,6\}$		

Table 2: Basic Probability Assignment, m, for Die Example

$m(\emptyset) = 0$	$m(\{1,2,3\}) = 0$	$m(\{1,2,3,5\}) = 0$
$m(\{1\}) = 0$	$m(\{1,2,4\}) = 0$	$m(\{1,2,3,6\}) = 0$
$m(\{2\}) = 0$	$m(\{1,2,5\}) = 0$	$m(\{1,2,4,5\}) = 0$
$m(\{3\}) = 0$	$m(\{1,2,6\}) = 0$	$m(\{1,2,4,6\}) = 0$
$m(\{4\}) = 0$	$m(\{1,3,4\}) = 0$	$m(\{1,2,5,6\}) = 0$
$m(\{5\}) = 0$	$m(\{1,3,5\}) = 0.5$	$m(\{1,3,4,5\}) = 0$
$m(\{6\}) = 0$	$m(\{1,3,6\}) = 0$	$m(\{1,3,4,6\}) = 0$
$m(\{1,2\}) = 0$	$m(\{1,4,5\}) = 0$	$m(\{1,3,5,6\}) = 0$
$m(\{1,3\}) = 0$	$m(\{1,4,6\}) = 0$	$m(\{1,4,5,6\}) = 0$
$m(\{1,4\}) = 0$	$m(\{1,5,6\}) = 0$	$m(\{2,3,4,5\}) = 0$
$m(\{1,5\}) = 0$	$m(\{2,3,4\}) = 0$	$m(\{2,3,4,6\}) = 0$
$m(\{1,6\}) = 0$	$m(\{2,3,5\}) = 0$	$m(\{2,3,5,6\}) = 0$
$m(\{2,3\}) = 0$	$m(\{2,3,6\}) = 0$	$m(\{2,4,5,6\}) = 0$
$m(\{2,4\}) = 0$	$m(\{2,4,5\}) = 0$	$m(\{3,4,5,6\}) = 0$
$m(\{2,5\}) = 0$	$m(\{2,4,6\}) = 0.5$	$m(\{1,2,3,4,5\}) = 0$
$m(\{2,6\}) = 0$	$m(\{2,5,6\}) = 0$	$m(\{1,2,3,4,6\}) = 0$
$m(\{3,4\}) = 0$	$m(\{3,4,5\}) = 0$	$m(\{1,2,4,5,6\}) = 0$
$m(\{3,5\}) = 0$	$m(\{3,4,6\}) = 0$	$m(\{1,2,3,5,6\}) = 0$
$m(\{3,6\}) = 0$	$m(\{3,5,6\}) = 0$	$m(\{1,3,4,5,6\}) = 0$
$m(\{4,5\}) = 0$	$m(\{4,5,6\}) = 0$	$m(\{2,3,4,5,6\}) = 0$
$m(\{4,6\}) = 0$	$m(\{1,2,3,4\}) = 0$	$m(\{1,2,3,4,5,6\}) = 0$
$m(\{5,6\}) = 0$		

Table 3: Belief for Die Example

$Bel(\emptyset) = 0$	$Bel(\{1,2,3\}) = 0$	$Bel(\{1,2,3,5\}) = 0.5$
$Bel(\{1\}) = 0$	$Bel(\{1,2,4\}) = 0$	$Bel(\{1,2,3,6\}) = 0$
$Bel(\{2\}) = 0$	$Bel(\{1,2,5\}) = 0$	$Bel(\{1,2,4,5\}) = 0$
$Bel(\{3\}) = 0$	$Bel(\{1,2,6\}) = 0$	$Bel(\{1,2,4,6\}) = 0.5$
$Bel(\{4\}) = 0$	$Bel(\{1,3,4\}) = 0$	$Bel(\{1,2,5,6\}) = 0$
$Bel(\{5\}) = 0$	$Bel(\{1,3,5\}) = 0.5$	$Bel(\{1,3,4,5\}) = 0.5$
$Bel(\{6\}) = 0$	$Bel(\{1,3,6\}) = 0$	$Bel(\{1,3,4,6\}) = 0$
$Bel(\{1,2\}) = 0$	$Bel(\{1,4,5\}) = 0$	$Bel(\{1,3,5,6\}) = 0.5$
$Bel(\{1,3\}) = 0$	$Bel(\{1,4,6\}) = 0$	$Bel(\{1,4,5,6\}) = 0$
$Bel(\{1,4\}) = 0$	$Bel(\{1,5,6\}) = 0$	$Bel(\{2,3,4,5\}) = 0$
$Bel(\{1,5\}) = 0$	$Bel(\{2,3,4\}) = 0$	$Bel(\{2,3,4,6\}) = 0.5$
$Bel(\{1,6\}) = 0$	$Bel(\{2,3,5\}) = 0$	$Bel(\{2,3,5,6\}) = 0$
$Bel(\{2,3\}) = 0$	$Bel(\{2,3,6\}) = 0$	$Bel(\{2,4,5,6\}) = 0.5$
$Bel(\{2,4\}) = 0$	$Bel(\{2,4,5\}) = 0$	$Bel(\{3,4,5,6\}) = 0$
$Bel(\{2,5\}) = 0$	$Bel(\{2,4,6\}) = 0.5$	$Bel(\{1,2,3,4,5\}) = 0.5$
$Bel(\{2,6\}) = 0$	$Bel(\{2,5,6\}) = 0$	$Bel(\{1,2,3,4,6\}) = 0.5$
$Bel(\{3,4\}) = 0$	$Bel(\{3,4,5\}) = 0$	$Bel(\{1,2,4,5,6\}) = 0.5$
$Bel(\{3,5\}) = 0$	$Bel(\{3,4,6\}) = 0$	$Bel(\{1,2,3,5,6\}) = 0.5$
$Bel(\{3,6\}) = 0$	$Bel(\{3,5,6\}) = 0$	$Bel(\{1,3,4,5,6\}) = 0.5$
$Bel(\{4,5\}) = 0$	$Bel(\{4,5,6\}) = 0$	$Bel(\{2,3,4,5,6\}) = 0.5$
$Bel(\{4,6\}) = 0$	$Bel(\{1,2,3,4\}) = 0$	$Bel(\{1,2,3,4,5,6\}) = 1.0$
$Bel(\{5,6\}) = 0$		

Table 4: Plausability for Die Example

$Pl(\emptyset) = 0$	$Pl(\{1,2,3\}) = 1.0$	$Pl(\{1,2,3,5\}) = 1.0$
$Pl(\{1\}) = 0.5$	$Pl(\{1,2,4\}) = 1.0$	$Pl(\{1,2,3,6\}) = 1.0$
$Pl(\{2\}) = 0.5$	$Pl(\{1,2,5\}) = 1.0$	$Pl(\{1,2,4,5\}) = 1.0$
$Pl(\{3\}) = 0.5$	$Pl(\{1,2,6\}) = 1.0$	$Pl(\{1,2,4,6\}) = 1.0$
$Pl(\{4\}) = 0.5$	$Pl(\{1,3,4\}) = 1.0$	$Pl(\{1,2,5,6\}) = 1.0$
$Pl(\{5\}) = 0.5$	$Pl(\{1,3,5\}) = 0.5$	$Pl(\{1,3,4,5\}) = 1.0$
$Pl(\{6\}) = 0.5$	$Pl(\{1,3,6\}) = 1.0$	$Pl(\{1,3,4,6\}) = 1.0$
$Pl(\{1,2\}) = 1.0$	$Pl(\{1,4,5\}) = 1.0$	$Pl(\{1,3,5,6\}) = 1.0$
$Pl(\{1,3\}) = 0.5$	$Pl(\{1,4,6\}) = 1.0$	$Pl(\{1,4,5,6\}) = 1.0$
$Pl(\{1,4\}) = 1.0$	$Pl(\{1,5,6\}) = 1.0$	$Pl(\{2,3,4,5\}) = 1.0$
$Pl(\{1,5\}) = 0.5$	$Pl(\{2,3,4\}) = 1.0$	$Pl(\{2,3,4,6\}) = 1.0$
$Pl(\{1,6\}) = 1.0$	$Pl(\{2,3,5\}) = 1.0$	$Pl(\{2,3,5,6\}) = 1.0$
$Pl(\{2,3\}) = 1.0$	$Pl(\{2,3,6\}) = 1.0$	$Pl(\{2,4,5,6\}) = 1.0$
$Pl(\{2,4\}) = 0.5$	$Pl(\{2,4,5\}) = 1.0$	$Pl(\{3,4,5,6\}) = 1.0$
$Pl(\{2,5\}) = 1.0$	$Pl(\{2,4,6\}) = 0.5$	$Pl(\{1,2,3,4,5\}) = 1.0$
$Pl(\{2,6\}) = 0.5$	$Pl(\{2,5,6\}) = 1.0$	$Pl(\{1,2,3,4,6\}) = 1.0$
$Pl(\{3,4\}) = 1.0$	$Pl(\{3,4,5\}) = 1.0$	$Pl(\{1,2,4,5,6\}) = 1.0$
$Pl(\{3,5\}) = 0.5$	$Pl(\{3,4,6\}) = 1.0$	$Pl(\{1,2,3,5,6\}) = 1.0$
$Pl(\{3,6\}) = 1.0$	$Pl(\{3,5,6\}) = 1.0$	$Pl(\{1,3,4,5,6\}) = 1.0$
$Pl(\{4,5\}) = 1.0$	$Pl(\{4,5,6\}) = 1.0$	$Pl(\{2,3,4,5,6\}) = 1.0$
$Pl(\{4,6\}) = 0.5$	$Pl(\{1,2,3,4\}) = 1.0$	$Pl(\{1,2,3,4,5,6\}) = 1.0$
$Pl(\{5,6\}) = 1.0$		

Table 5: Basic Probability Assignment, Belief, and Plausability for Reliability Example

	$m()$	$Bel()$	$Pl()$
$\emptyset$	0	0	0
{survives to t}	$1 - p_a$	$1 - p_a$	1
{fails by t}	0	0	p_a
{survives to t, fails by t}	p_a	1	1

Table 6: States of Nature, Actions, and Utilities for Reliability Example

	Use at t	Do not use at t
{survives to t}	2	-1
{fails by t}	-5	1